Math 304 Sample Final

Name: ______________________

This exam has 11 questions, for a total of 150 points.

Please answer each question in the space provided. Please write **full solutions**, not just answers. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	10	
2	15	
3	10	
4	10	
5	10	
6	20	
7	20	
8	15	
9	10	
10	15	
11	15	
Total:	150	

Question 1. (10 pts)

Determine the following statements are true or false.

(a) If B is diagonalizable, then B^2 is also diagonalizable.

Solution: True.

(b) If $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ span $\mathbb{R}^8$, then n must be 8.

Solution: False.

(c) If $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linear independent in $\mathbb{R}^7$, then n must be at most 7.

Solution: True.

(d) Suppose that A is a diagonalizable 5×5 matrix. If $A^5 = 0$, then A must be the zero matrix.

Solution: True.

(e) Suppose $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ form a basis of $\mathbb{R}^3$, then $\vec{v}_1$, $(\vec{v}_1 + \vec{v}_2)$ and $(\vec{v}_1 + \vec{v}_2 + \vec{v}_3)$ also form a basis of $\mathbb{R}^3$.

Solution: True.

Question 2. (15 pts)

(a) Let A be a (4×4) matrix. We view A as a linear mapping $\mathbb{R}^4 \to \mathbb{R}^4$. Suppose $\det(A) \neq 0$. What is the dimension of the range of A? Justify your answer.

Solution: $\det A \neq 0$ implies that $\operatorname{Ker} A = \{0\}$. We know that

$$\dim \mathbb{R}^4 = \dim \operatorname{Im} A + \dim \operatorname{Ker} A.$$

Therefore,

$$\dim \operatorname{Im} A = 4 - 0 = 4$$

(b) Suppose $F : \mathbb{R}^5 \to \mathbb{R}^3$ is linear mapping. Is it possible that $\dim \ker F = 0$? Justify your answer.

Solution: We have the following formula

$$\dim R^5 = \dim \operatorname{Ker} F + \dim \operatorname{Im} F$$

Note that $\dim \operatorname{Im} F$ is a subspace of $\mathbb{R}^3$. In particular,

$$\dim \operatorname{Im} F \leq 3.$$

We also know that

$$\dim R^5 = \dim \operatorname{Im} F + \dim \operatorname{Ker} F.$$

This implies that $\dim \operatorname{Ker} F = 5 - \dim \operatorname{Im} F \geq 2$. So F cannot be injective.

(c) Let $B : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping. What are the possible integer values that $\dim(\ker B)$ can take? List all possibilities.

Solution: All possible values of $\dim(\ker B)$ are 0, 1 and 2.

Question 3. (10 pts)

Find the determinant of

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution: It is a standard computation. I skip the details. The determinant is -1.

Question 4. (10 pts)

Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 2-i \\ 2+i & 0 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2-i \\ 2+i & -\lambda \end{vmatrix} = -\lambda(4-\lambda) - 5 = (\lambda - 5)(\lambda + 1)$$

When $\lambda = 5$, the eigenvector is

$$v = \begin{bmatrix} (2-i) \\ 1 \end{bmatrix}$$

When $\lambda = -1$, the eigenvector is

$$w = \begin{bmatrix} -1 \\ (2+i) \end{bmatrix}$$

Question 5. (10 pts)

Given the vectors

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 5 \\ 8 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^4$. Determined whether $v = (3, 0, 1, 4)^T$ is in the span of u_1, u_2, u_3.

Solution: Form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 5 & 0 \\ 1 & 3 & 8 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Its row echelon form is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

So the system is inconsistent. So v is not in the span of u_1, u_2, u_3.

Question 6. (20 pts)

Given the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

(a) Find the reduced row echelon form of A.

Solution: I skip the details. The row echelon form is

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find a basis for the kernel of A.

Solution: All elements in the kernel are of the form

$$s\begin{pmatrix}-4\\1\\0\\1\end{pmatrix}+t\begin{pmatrix}-2\\3\\1\\0\end{pmatrix}$$

So

$$v_1=\begin{pmatrix}-4\\1\\0\\1\end{pmatrix},\ v_2=\begin{pmatrix}-2\\3\\1\\0\end{pmatrix}$$

form a basis of $\ker A$.

(c) Find a basis for the range of A.

Solution: The first and second columns contain pivotal entries. So the corresponding columns in A form a basis of the range of A, that is,

$$\begin{pmatrix}1\\0\\3\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\4\\-1\end{pmatrix}$$

Question 7. (20 pts)

Determine whether the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is diagonalizable. If yes, find a nonsingular matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

Solution: Use cofactor expansion along the first column

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = (1-\lambda)\lambda^2$$

So $\lambda = 1$ and 0.

When $\lambda = 1$, solve for the kernel of the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We find an eigenvector $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

When $\lambda = 0$, solve for the kernel of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We find an eigenvector $w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

We can verify that v, w_1 and w_2 are linearly independent. It follows that A has 3 linearly independent eigenvectors, so A is diagonalizable.

Let

$$S = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $A = SDS^{-1}$.

Question 8. (15 pts)

Let V be the subspace of $\mathbb{R}^4$ spanned by

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 4 \end{bmatrix}.$$

(a) Use Gram-Schmidt process to find an orthonormal basis of V.

Solution: I skip the details.

$$u_1 = \frac{v_1}{\|v_1\|} = (0, 1, 0, 0)^T$$

$$u_2 = (\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}})^T$$

$$u_3 = (0, 0, 1, 0)^T$$

(b) Find the projection of

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

onto V.

Solution: The projection is

$$\langle \vec{x}, u_1 \rangle u_1 + \langle \vec{x}, u_2 \rangle u_2 + \langle \vec{x}, u_3 \rangle u_3 = \begin{bmatrix} 3/2 \\ -1 \\ 5 \\ 3/2 \end{bmatrix}$$

Question 9. (10 pts)

Let $\mathbb{P}_2 = \{$all polynomials of degree $\leq 2\}$. We define the following inner product on $\mathbb{P}_2$:

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Find the angle between $x+1$ and x^2.

Solution:

$$\|x+1\|^2 = \int_{-1}^{1} (x+1)^2 dx = \frac{8}{3}$$

So $\|x+1\| = \sqrt{\frac{8}{3}}$. Similarly, we compute $\|x^2\| = \sqrt{\frac{2}{5}}$.

$$\langle x+1, x^2 \rangle = \frac{2}{3}$$

So

$$\cos\theta = \frac{\langle x+1, x^2 \rangle}{\|x+1\|\|x^2\|} = \frac{\sqrt{15}}{6}$$

The angle is

$$\theta = \arccos(\frac{\sqrt{15}}{6})$$

Question 10. (15 pts)

Let $V = \text{span}(e^x, e^{-x}, xe^x, xe^{-x})$ be a subspace of C^∞. We know that V is a vector space with a basis

$$e^x,\ e^{-x},\ xe^x,\ xe^{-x}.$$

Let us denote this basis by $\mathfrak{B}$. Let

$$T(f) = 2f - f'$$

be a linear transformation from V to V .

(a) Find the matrix representation of T with the basis $\mathfrak{B}$.

Solution: The matrix is

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(b) Is T nonsingular? Justify your answer.

Solution:

$$\det(A) = 9 \neq 0$$

So T is nonsingular.

Question 11. (15 pts)

The eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ are $\lambda_1 = 1$ with $v_1 = (1,0,0)^T$, $\lambda_2 = 2$ with $v_2 = (1,1,0)^T$ and $\lambda_3 = 3$ with $v_3 = (1,2,1)^T$.

(a) Find the general solution to the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

In other words,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ c_2 e^{2t} + 2c_3 e^{3t} \\ c_3 e^{3t} \end{bmatrix}$$

(b) Find a specific solution $\mathbf{x}(t)$ such that

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

when $t = 0$.

Solution:

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + 2c_3 \\ c_3 \end{bmatrix}.$$

Therefore, we need to solve for c_1, c_2 and c_3 of the following linear system

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + 2c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

which has a unique solution $c_1 = 3, c_2 = 2$ and $c_3 = 1$. So the solution satisfying the given initial condition is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 3e^t + 2e^{2t} + e^{3t} \\ 2e^{2t} + 2e^{3t} \\ e^{3t} \end{bmatrix}$$